NOTE ON MMAT 5010: LINEAR ANALYSIS (2019 1ST TERM)

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1. Lecture 1

Throughout this note, we always denote $\mathbb{K}$ by the real field $\mathbb{R}$ or the complex field $\mathbb{C}$. Let $\mathbb{N}$ be the set of all natural numbers. Also, we write a sequence of numbers as a function $x : \{1, 2, ...\} \to \mathbb{K}$ or $x_i := x(i)$ for $i = 1, 2....$

Definition 1.1. *Let X be a vector space over the field $\mathbb{K}$. A function $\|\cdot\| : X \to \mathbb{R}$ is called a norm on X if it satisfies the following conditions.*

(i) $\|x\| \geq 0$ for all $x \in X$ and $\|x\| = 0$ if and only if $x = 0$.
(ii) $\|\alpha x\| = |\alpha|\|x\|$ for all $\alpha \in \mathbb{K}$ and $x \in X$.
(iii) $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in X$.

In this case, the pair $(X, \|\cdot\|)$ is called a normed space.

Remark 1.2. *Recall that a* metric space *is a non-empty set Z together with a function, (*called a metric*), $d : Z \times Z \to \mathbb{R}$ that satisfies the following conditions:*

(i) $d(x, y) \geq 0$ for all $x, y \in Z$; and $d(x, y) = 0$ if and only if $x = y$.
(ii) $d(x, y) = d(y, x)$ for all $x, y \in Z$.
(iii) $d(x, y) \leq d(x, z) + d(z, y)$ for all x, y and z in Z.

For a normed space $(X, \|\cdot\|)$, if we define $d(x, y) := \|x - y\|$ for $x, y \in X$, then X becomes a metric space under the metric d.

The following examples are important classes in the study of functional analysis.

Example 1.3. *Consider $X = \mathbb{K}^n$. Put*

$$\|x\|_p := \big(\sum_{i=1}^{n} |x_i|^p\big)^{1/p} \quad \textit{and} \quad \|x\|_\infty := \max_{i=1,...,n} |x_i|$$

for $1 \leq p < \infty$ and $x = (x_1, ..., x_n) \in \mathbb{K}^n$.
Then $\|\cdot\|_p$ (called the usual norm as p=2) and $\|\cdot\|_\infty$ (called the sup*-norm) all are norms on $\mathbb{K}^n$.*

Example 1.4. *Put*

$$c_0 := \{(x(i)) : x(i) \in \mathbb{K}, \ \lim |x(i)| = 0\} \textit{(called the null sequnce space)}$$

and

$$\ell^\infty := \{(x(i)) : x(i) \in \mathbb{K}, \ \sup_i |x(i)| < \infty\}.$$

Then c_0 is a subspace of ℓ^∞. The sup*-norm $\|\cdot\|_\infty$ on ℓ^∞ is defined by*

$$\|x\|_\infty := \sup_i |x(i)|$$

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for $x \in \ell^\infty$. *Let*

$$c_{00} := \{(x(i)) : \text{ there are only finitly many } x(i)\text{'s are non-zero}\}.$$

Also, c_{00} *is endowed with the* sup*-norm defined above and is called the finite sequence space.*

Example 1.5. *For* $1 \le p < \infty$*, put*

$$\ell^p := \{(x(i)) : x(i) \in \mathbb{K}, \ \sum_{i=1}^{\infty} |x(i)|^p < \infty\}.$$

Also, ℓ^p *is equipped with the norm*

$$\|x\|_p := (\sum_{i=1}^{\infty} |x(i)|^p)^{\frac{1}{p}}$$

for $x \in \ell^p$*. Then* $\|\cdot\|_p$ *is a norm on* ℓ^p *(see* [2, Section 9.1]*).*

Example 1.6. *Let* $C^b(\mathbb{R})$ *be the space of all bounded continuous* $\mathbb{R}$*-valued functions* f *on* $\mathbb{R}$*. Now* $C^b(\mathbb{R})$ *is endowed with the* sup*-norm, that is,*

$$\|f\|_\infty = \sup_{x\in\mathbb{R}} |f(x)|$$

for every $f \in C^b(\mathbb{R})$*. Then* $\|\cdot\|_\infty$ *is a norm on* $C^b(\mathbb{R})$*.*
Also, we consider the following subspaces of $C^b(X)$*.*
Let $C_0(\mathbb{R})$ $\big($*resp.* $C_c(\mathbb{R})\big)$ *be the space of all continuous* $\mathbb{R}$*-valued functions* f *on* $\mathbb{R}$ *which vanish at infinity (resp. have compact supports), that is, for every* $\varepsilon > 0$*, there is a* $K > 0$ *such that* $|f(x)| < \varepsilon$ *(resp.* $f(x) \equiv 0$*) for all* $|x| > K$*.*
It is clear that we have $C_c(\mathbb{R}) \subseteq C_0(\mathbb{R}) \subseteq C^b(\mathbb{R})$*.*
Now $C_0(\mathbb{R})$ *and* $C_c(\mathbb{R})$ *are endowed with the* sup*-norm* $\|\cdot\|_\infty$*.*

From now on, we always let X be a normed sapce.

Definition 1.7. *We say that a sequence* (x_n) *in* X *converges to an element* $a \in X$ *if* $\lim \|x_n - a\| = 0$*, that is, for any* $\varepsilon > 0$*, there is* $N \in \mathbb{N}$ *such that* $\|x_n - a\| < \varepsilon$ *for all* $n \ge N$*.*
In this case, (x_n) *is said to be convergent and* a *is called a limit of the sequence* (x_n)*.*

Remark 1.8.
(i) *If* (x_n) *is a convergence sequence in* X*, then its limit is unique. In fact, if* a *and* b *both are the limits of* (x_n)*, then we have* $\|a-b\| \le \|a-x_n\|+\|x_n-b\| \to 0$*. So,* $\|a-b\| = 0$ *which implies that* $a = b$*.*

We write $\lim x_n$ **for the limit of** (x_n) **provided the limit exists.**

(ii) *The definition of a convergent sequence* (x_n) *depends on the underling space where the sequence* (x_n) *sits in. For example, for each* $n = 1, 2...$*, let* $x_n(i) := 1/i$ *as* $1 \le i \le n$ *and* $x_n(i) = 0$ *as* $i > n$*. Then* (x_n) *is a convergent sequence in* ℓ^∞ *but it is not convergent in* c_{00}*.*

The following is one of the basic properties of a normed space. The proof is directly shown by the triangle inequality and a simple fact that every convergent sequence (x_n) must be *bounded*, i.e., there is a positive number M such that $\|x_n\| \le M$ for all $n = 1, 2, ...$

Proposition 1.9. *The addition* $+ : (x, y) \in X \times X \mapsto x + y \in X$ *and the scalar multiplication* $\bullet : (\lambda, x) \in \mathbb{K} \times X \mapsto \lambda x \in X$ *both are continuous maps. More precisely, if the convergent sequences* $x_n \to x$ *and* $y_n \to y$ *in* X*, then we have* $x_n + y_n \to x + y$*. Similarly, if a sequence of numbers* $\lambda_n \to \lambda$ *in* $\mathbb{K}$*, then we also have* $\lambda_n x_n \to \lambda x$*.*

A sequence (x_n) in X is called a ***Cauchy sequence*** if for any $\varepsilon > 0$, there is $N \in \mathbb{N}$ such that $\|x_m - x_n\| < \varepsilon$ for all $m, n \geq N$. We have the following simple observation.

Proposition 1.10. *Every convergent sequence in* X *is a Cauchy sequence.*

Proof. Let (x_n) be a convergent sequence with the limit a in X. Then for any $\varepsilon > 0$, there is a positive integer N such that $\|x_n - a\| < \varepsilon$ for all $n \geq N$. This implies that $\|x_m - x_n\| \leq \|x_n - a\| + \|a - x_m\| < 2\varepsilon$ for all $m, n \geq N$. Thus, (x_n) is a Cauchy sequence. □

Remark 1.11. *The converse of Proposition 1.10 does not hold.*
For example, let X *be the finite sequence space* $(c_{00}, \|\cdot\|_\infty)$*. If we consider the sequence* $x_n := (1, 1/2, 1/3, ..., 1/n, 0, 0, ...) \in c_{00}$*, then* (x_n) *is a Cauchy sequence but it is not a convergent sequence in* c_{00}*.*
In fact, if we are given any element $a \in c_{00}$*, then there exists a positive integer* N *such that* $a(i) = 0$ *for all* $i \geq N$*. Thus we always have* $\|x_n - a\|_\infty \geq 1/N$ *for all* $n \geq N$ *and thus,* $\|x_n - a\|_\infty \nrightarrow 0$*. This implies that the sequence* (x_n) *does not converge to any element in* c_{00}*.*

The following notation plays an important role in mathematics.

Definition 1.12. *A normed space* X *is said to be a Banach space if every Cauchy sequence in* X *must be convergent. The space* X *is also said to be complete in this case.*

Example 1.13. *With the notation as above, we have the following examples of Banach spaces.*

(i) *If* $\mathbb{K}^n$ *is equipped with the usual norm, then* $\mathbb{K}^n$ *is a Banach space.*
(ii) ℓ^∞ *is a Banach space. In fact, if* (x_n) *is a Cauchy sequence in* ℓ^∞*, then for any* $\varepsilon > 0$*, there is* $N \in \mathbb{N}$*, we have*

$$|x_n(i) - x_m(i)| \leq \|x_n - x_m\|_\infty < \varepsilon$$

for all $m, n \geq N$ *and* $i = 1, 2.....$ *Thus, if we fix* $i = 1, 2, ..,$ *then* $(x_n(i))_{n=1}^\infty$ *is a Cauchy sequence in* $\mathbb{K}$*. Since* $\mathbb{K}$ *is complete, the limit* $\lim_n x_n(i)$ *exists in* $\mathbb{K}$ *for all* $i = 1, 2....$ *Nor for each* $i = 1, 2...,$ *we put* $z(i) := \lim_n x_n(i) \in \mathbb{K}$*. Then we have* $z \in \ell^\infty$ *and* $\|z - x_n\|_\infty \to 0$*. So,* $\lim_n x_n = z \in \ell^\infty$ *(***Check !!!!***). Thus* ℓ^∞ *is a Banach space.*
(iii) ℓ^p *is a Banach space for* $1 \leq p < \infty$*. The proof is similar to the case of* ℓ^∞*.*
(iv) $C[a, b]$ *is a Banach space.*
(v) *Let* $C_0(\mathbb{R})$ *be the space of all continuous* $\mathbb{R}$*-valued functions* f *on* $\mathbb{R}$ *which are vanish at infinity, that is, for every* $\varepsilon > 0$*, there is a* $M > 0$ *such that* $|f(x)| < \varepsilon$ *for all* $|x| > M$*. Now* $C_0(\mathbb{R})$ *is endowed with the* sup*-norm, that is,*

$$\|f\|_\infty = \sup_{x \in \mathbb{R}} |f(x)|$$

for every $f \in C_0(\mathbb{R})$*. Then* $C_0(\mathbb{R})$ *is a Banach space.*

Notation 1.14. *For* $r > 0$ *and* $x \in X$*, let*

(i) $B(x, r) := \{y \in X : \|x - y\| < r\}$ *(called an open ball with the center at* x *of radius* r*) and* $B^*(x, r) := \{y \in X : 0 < \|x - y\| < r\}$

(ii) $B(x,r) := \{y \in X : \|x-y\| \leq r\}$ *(called a closed ball with the center at x of radius r).*

Put $B_X := \{x \in X : \|x\| \leq 1\}$ *and* $S_X := \{x \in X : \|x\| = 1\}$ *the closed unit ball and the unit sphere of X respectively.*

Definition 1.15. *Let A be a subset of X.*

(i) A point $a \in A$ is called an interior point of A if there is $r > 0$ such that $B(a,r) \subseteq A$. Write $int(A)$ for the set of all interior points of A.
(ii) A is called an open subset of X if $int(A) = A$.

Example 1.16. *We keep the notation as above.*

(i) Let $\mathbb{Z}$ and $\mathbb{Q}$ denote the set of all integers and rational numbers respectively If $\mathbb{Z}$ and $\mathbb{Q}$ both are viewed as the subsets of $\mathbb{R}$, then $int(\mathbb{Z})$ and $int(\mathbb{Q})$ both are empty.
(ii) The open interval $(0,1)$ is an open subset of $\mathbb{R}$ but it is not an open subset of $\mathbb{R}^2$. In fact, $int(0,1) = (0,1)$ if $(0,1)$ is considered as a subset of $\mathbb{R}$ but $int(0,1) = \emptyset$ while $(0,1)$ is viewed as a subset of $\mathbb{R}^2$.
*(iii) Every open ball is an open subset of X (**Check!!**).*

Definition 1.17. *Let A be a subset of X.*

(i) A point $z \in X$ is called a limit point of A if for any $\varepsilon > 0$, there is an element $a \in A$ such that $0 < \|z-a\| < \varepsilon$, that is, $B^*(z,\varepsilon) \cap A \neq \emptyset$ for all $\varepsilon > 0$.
Furthermore, if A contains the set of all its limit points, then A is said to be closed in X.
(ii) The closure of A, write $\overline{A}$, is defined by

$$\overline{A} := A \cup \{z \in X : z \text{ is a limit point of } A\}.$$

Remark 1.18. *With the notation as above, it is clear that a point $z \in \overline{A}$ if and only if $B(z,r) \cap A \neq \emptyset$ for all $r > 0$. This is also equivalent to saying that there is a sequence (x_n) in A such that $x_n \to a$. In fact, this can be shown by considering $r = \frac{1}{n}$ for $n = 1, 2....$*

Proposition 1.19. *With the notation as before, we have the following assertions.*

(i) A is closed in X if and only if its complement $X \setminus A$ is open in X.
(ii) The closure $\overline{A}$ is the smallest closed subset of X containing A. The "smallest" in here means that if F is a closed subset containing A, then $\overline{A} \subseteq F$.
Consequently, A is closed if and only if $\overline{A} = A$.

Proof. If A is empty, then the assertions (i) and (ii) both are obvious. Now assume that $A \neq \emptyset$. For part (i), let $C = X \setminus A$ and $b \in C$. Suppose that A is closed in X. If there exists an element $b \in C \setminus int(C)$, then $B(b,r) \subsetneqq C$ for all $r > 0$. This implies that $B(b,r) \cap A \neq \emptyset$ for all $r > 0$ and hence, b is a limit point of A since $b \notin A$. It contradicts to the closeness of A. So, $C = int(C)$ and thus, C is open.
For the converse of (i), assume that C is open in X. Assume that A has a limit point z but $z \notin A$. Since $z \notin A$, $z \in C = int(C)$ because C is open. Hence, we can find $r > 0$ such that $B(z,r) \subseteq C$. This gives $B(z,r) \cap A = \emptyset$. This contradicts to the assumption of z being a limit point of A. So, A must contain all of its limit points and hence, it is closed.

For part (ii), we first claim that $\overline{A}$ is closed. Let z be a limit point of $\overline{A}$. Let $r > 0$. Then there is $w \in B^*(z,r) \cap \overline{A}$. Choose $0 < r_1 < r$ small enough such that $B(w,r_1) \subseteq B^*(z,r)$. Since w is a limit point of A, we have $\emptyset \neq B^*(w,r_1) \cap A \subseteq B^*(z,r) \cap A$. So, z is a limit point of A. Thus, $z \in \overline{A}$ as required. This implies that $\overline{A}$ is closed.
It is clear that $\overline{A}$ is the smallest closed set containing A.
The last assertion follows from the minimality of the closed sets containing A immediately.
The proof is finished. □

Example 1.20. *Retains all notation as above. We have* $\overline{c_{00}} = c_0 \subseteq \ell^\infty$. *Consequently,* c_0 *is a closed subspace of* ℓ^∞ *but* c_{00} *is not.*

Proof. We first claim that $\overline{c_{00}} \subseteq c_0$. Let $z \in \ell^\infty$. It suffices to show that if $z \in \overline{c_{00}}$, then $z \in c_0$, that is, $\lim_{i\to\infty} z(i) = 0$. Let $\varepsilon > 0$. Then there is $x \in B(z, \varepsilon) \cap c_{00}$ and hence, we have $|x(i) - z(i)| < \varepsilon$ for all $i = 1, 2.....$ Since $x \in c_{00}$, there is $i_0 \in \mathbb{N}$ such that $x(i) = 0$ for all $i \geq i_0$. Therefore, we have $|z(i)| = |z(i) - x(i)| < \varepsilon$ for all $i \geq i_0$. So, $z \in c_0$ as desired.

For the reverse inclusion, let $w \in c_0$. It needs to show that $B(w, r) \cap c_{00} \neq \emptyset$ for all $r > 0$. Let $r > 0$. Since $w \in c_0$, there is i_0 such that $|w(i)| < r$ for all $i \geq i_0$. If we let $x(i) = w(i)$ for $1 \leq i < i_0$ and $x(i) = 0$ for $i \geq i_0$, then $x \in c_{00}$ and $\|x - w\|_\infty := \sup_{i=1,2...} |x(i) - w(i)| < r$ as required. □

Proposition 1.21. *Let* Y *be a subspace of a Banach space* X*. Then* Y *is a Banach space if and only if* Y *is closed in* X*.*

Proof. For the necessary condition, we assume that Y is a Banach space. Let $z \in \overline{Y}$. Then there is a convergent sequence (y_n) in Y such that $y_n \to z$. Since (y_n) is convergent, it is also a Cauchy sequence in Y. Then (y_n) is also a convergent sequence in Y because Y is a Banach space. So, $z \in Y$. This implies that $\overline{Y} = Y$ and hence, Y is closed.

For the converse statement, assume that Y is closed. Let (z_n) be a Cauchy sequence in Y. Then it is also a Cauchy sequence in X. Since X is complete, $z := \lim z_n$ exists in X. Note that $z \in Y$ because Y is closed. So, (z_n) is convergent in Y. Thus, Y is complete as desired. □

Corollary 1.22. c_0 *is a Banach space but the finite sequence* c_{00} *is not.*

Proposition 1.23. *Let* $(X, \|\cdot\|)$ *be a normed space. Then there is a normed space* $(X_0, \|\cdot\|_0)$*, together with a linear map* $i : X \to X_0$*, satisfy the following condition.*

(i) X_0 *is a Banach space.*
(ii) *The map* i *is an isometry, that is,* $\|i(x)\|_0 = \|x\|$ *for all* $x \in X$*.*
(iii) *the image* $i(X)$ *is dense in* X_0*, that is,* $\overline{i(X)} = X_0$*.*

Moreover, such pair (X_0, i) *is unique up to isometric isomorphism in the following sense: if* $(W, \|\cdot\|_1)$ *is a Banach space and an isometry* $j : X \to W$ *is an isometry such that* $\overline{j(X)} = W$*, then there is an isometric isomorphism* ψ *from* X_0 *onto* W *such that*

$$j = \psi \circ i : X \to X_0 \to W.$$

In this case, the pair (X_0, i) *is called the completion of* X*.*

Example 1.24. *Proposition 1.23 cannot give an explicit form of the completion of a given normed space. The following examples are basically due to the uniqueness of the completion.*

(i) *If* X *is a Banach space, then the completion of* X *is itself.*
(ii) *By Corollary 1.22, the completion of the finite sequence space* c_{00} *is the null sequence space* c_0*.*
(iii) *The completion of* $C_c(\mathbb{R})$ *is* $C_0(\mathbb{R})$*.*

References

[1] Introductory functional analysis with applications, Wiley, (1989).
[2] J. Muscat, Functional analysis, Springer, (2014).